An Integrated Approach to Structural Synthesis and Analysis

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The structural synthesis problem, which is currently viewed as an inequality constrained minimization problem in design variable space, is transformed into an unconstrained minimization problem in a space such that each point represents a set of values for the design variables as well as for each behavior variable in each load condition. Using penalty functions and a trial value of the optimum weight (W_o), rather than slack variables and Lagrange multipliers, a positive function ψ is constructed such that any point where $\psi = 0$ represents an acceptable design of weight W_o or less. This integrated formulation is used to seek optimum designs by systematically reducing W_o, thus generating a sequence of acceptable designs with decreasing weight. Numerical examples of optimum designs, based on a nonlinear analysis for a one node m-bar planar truss, are presented together with computer running times. These results are obtained using a steep-descent-type procedure in conjunction with the ψ-function formulation. The results indicate that the integrated approach offers the prospect of making substantial improvements in the efficiency of the structural synthesis process, particularly when linearization of the structural analysis is inappropriate.

Nomenclature

$\bar{A}_i$ = cross-sectional area of the ith member‡
$\bar{A}_{io}$ = constant of the order of $\bar{A}_i$
BND_j = simple bound penalty function of the variable x_j
$[BF_{ij}]$ = array of behavior functions
C_{ik} = notation defined by Eq. (27)
$(D/t)_i$ = mean diameter-to-thickness ratio for the ith annular member
$\{D_p\}$ = vector containing the design variables
$\{D_p^{(U)}\}$, $\{D_p^{(L)}\}$ = upper and lower limits on $\{D_p\}$
$\bar{d}_i$ = position of the attachment points
E_i = Youngs modulus of the ith member
EQX_k, EQY_k = normalized equilibrium residuals in the x and y directions for the kth load condition
FW = normalized weight penalty function
G_{ik} = buckling function, see Eq. (48)
g = percentage by which the buckling critical stress may be violated
H = height of the truss
h_q = qth move distance in minimization method
K_i = Ramberg-Osgood constant for the ith member
K_A, K_σ, K_u = nondimensionalizing constants, see Eqs. (15–19)
k = integer indicating the load condition, $k = 1, \ldots, n$
LB_j = lower bounds on x_j
$\overline{LD}_{ik}$ = length of the member i after deformation by the kth load condition
$\overline{LDX}_{ik}$, $\overline{LDY}_{ik}$ = X and Y projections of $\overline{LD}_{ik}$
$[L_{ij}]$ = lower limits on $[BF_{ij}]$
$\overline{LO}_i$ = original length of the ith member
l = integer denoting the total number of variables, see Eq. (45)
m = number of members in the single-node truss
M_i = Ramberg-Osgood exponent for the ith member, M_i is a positive, odd integer
n = number of load conditions
P_k = magnitude of the kth load
PX_k = X component of P_k
PY_k = Y component of P_k
$\bar{P}$ = average of the magnitudes of the nonzero P_k
$\hat{P}$ = constant, see Eq. (35)
Q_j = amount by which the x_j may violate either UB_j or LB_j
R_i = constant, see Eq. (31)
SD_{ik} = normalized stress-displacement residual
SF_{ik} = required safety factor on buckling stress in the ith member in the kth load condition
S_{ik} = notation defined by Eq. (28)
s = constant small compared to 1
TB_{ik} = tangent modulus buckling constraint penalty function
UB_j = upper bound on x_j
$[U_{ij}]$ = upper limits on $[BF_{ij}]$
$\bar{u}_k$ = x component of displacement of the node in the kth load condition
$\bar{v}_k$ = y component of displacement of the node in the kth load condition
W = weight of the structure
W_o = draw-down weight
$\tilde{W}$ = weight obtained at the completion of a convergence of $\psi \rightarrow 0$
x_j = substitute name for all variables, see Eqs. (40–44)
$\mathbf{x}$ = vector of x_j
$\mathbf{x}^q$ = qth vector $\mathbf{x}$
Y_i = yield stress of the ith member
α_k = angle from horizontal at which P_k is applied
$\bar{\alpha}_i$ = thermal expansion coefficient in the ith member
β_{ik} = angle between the horizontal and the ith member in the kth load condition
ΔT_{ik} = temperature change in the ith member in the kth load condition
Δ = a number less than one, the draw-down increment
δ_i = constant, see Eq. (38)
ϵ = constant, small compared to 1, the criterion of convergence of the residuals
η_i = constant, see Eq. (32)
θ_1 = hypothetical residual function
θ_2 = hypothetical residual function
θ_3 = θ_2 + penalty functions
λ_w, λ_j, $\lambda_{TB_{ik}}$ = constants, see Eqs. (53, 58, and 62)
ρ_i = weight density of the ith member
$\bar{\sigma}_{ik}$ = stress in the ith member in the kth load condition, tensile stress is positive
τ_{ik} = constant, see Eq. (33)
ψ = integrated synthesis-analysis function
∇ = gradient operator
$< >^n$ = bracket function defined by Eqs. (4) and (5)

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‡ All barred symbols, when used without the overbar, are nondimensional, as given by Eqs. (15–19, 24–26, and 34).

Introduction

STRUCTURAL synthesis has been defined as the rational, directed evolution of a structural design, which, in terms of a defined criterion, efficiently performs a set of specified functional purposes.[1] Considerable effort is being directed toward the development of rational automatable methods of structural design. Such methods will make it possible to more fully exploit the extensive and growing body of experimental and analytical knowledge in terms of superior engineering designs. Therefore, the central objective of structural-synthesis research is to bring an ever more meaningful class of structural-design problems within the grasp of rational, automated optimization in terms of realistic criteria. Advances in digital-computer technology, which give greater speed and more extensive storage capacity, serve this end. However, it is clear that innovations in the formulation of synthesis problems and improvements in mathematical programing techniques serve to expand the scope of design problems for which an automated optimum design capability is attainable within the framework of any fixed digital-computer capacity.

This paper reports the results of an investigation of a new approach to structural synthesis and analysis, which makes it possible to find an optimum design and its behavior without engaging in the evaluation of a large number of intermediate trial designs. The structural-synthesis problem is formulated so that optimum designs can be found by the application of any technique for seeking the unconstrained minimum of a function of many variables. Formulating the structural-synthesis task as an unconstrained minimization problem makes it possible to draw on a wide variety of techniques found in the literature in Appendix A of Ref. 2. This formulation, hereafter referred to as the integrated formulation, is characterized by the fact that the quest for an improved design and the search for values of the behavior variables are carried on simultaneously. The integrated formulation is also shown to facilitate the use of nonlinear analysis.

It should be noted at this point that the formulation of a structural-synthesis problem involves the exercise of engineering judgment in several ways. The adoption of a design philosophy and an optimization criterion are fundamental judgments that must be rendered prior to developing a structural-synthesis capability. An example of these judgments is seen in Ref. 9 where a weight minimization technique is developed which includes sizing and configuration variables for truss structures. This technique is based upon a limit design philosophy, which, of course, is one of many possible choices. The replacement of an environment envelope by a set of distinct load conditions is an idealization traditionally employed by structural engineers. Analysis techniques are employed to predict the behavior of proposed designs and guard against unacceptable behavior in each load condition. Structural failures can usually be ascribed to the omission of a critical load condition or an active failure mode from the design process. Undertaking the formulation of a synthesis problem often helps to point up specific needs for improved analyses and/or loading information. Structural-synthesis research can be effectively pursued by taking the position that the best load information and analysis techniques, consistent with the design goals, should be used in developing synthesis capabilities. Engineering judgments that yield a design philosophy and an optimization criterion, as well as suitable loading information and analysis techniques, are a prerequisite to undertaking the development of a structural-synthesis capability.

It has been shown previously[3] that the determination of a minimum-weight-balanced optimum design for a structural system subject to a multiplicity of load conditions can be treated as a mathematical programing problem. The block diagram in Fig. 1 outlines the usual design-cycle process employed in seeking a solution to the minimum-weight optimum-design problem. Previous structural synthesis efforts[4, 5] have been based on automating the design-cycle process illustrated in Fig. 1.

In Refs. 4 and 5 the redesign phase has been handled using various design-cycle-based techniques. In the design space, each point D_p represents a design, and the space is separated into an acceptable and an unacceptable region by the composite constraint surface, which may contain contributions from any of the limitations imposed. The redesign problem viewed in the design space may be characterized by the following questions: 1) In what direction should a move be made? 2) How long a step should be taken in that direction? 3) Is the current point a local minimum? 4) Is the current local minimum a global minimum?

Attempts to answer these questions, in any given redesign step, often require the analysis of many proposed designs before an improved acceptable design is found. Thus it is clear that the design-cycle-based methods gather a large amount of information by making complete analyses of many trial designs in the course of evolving a synthesis path terminating at a proposed optimum design. Most of this information is used solely to decide whether or not proposed trial designs are acceptable. One approach to improving the efficiency of these synthesis procedures would be to make more extensive use of this wealth of information. However, using the information to achieve a more highly directed synthesis technique, as well as gathering it in the first place, consumes substantial amounts of machine time. The growing importance of including nonlinearities in the structural analyses on which optimizations are to be based, further emphasizes the desirability of eliminating the time-consuming analysis of large numbers of individual trial designs. The approach to the structural-synthesis problem reported here converts the problem to one of finding the unconstrained minimum of a function of several variables and eliminates the design cycle in the conventional sense. It is noteworthy that analyses are accomplished only for designs that are acceptable and which offer a specified weight reduction.

Integrated Synthesis-Analysis Concept

It is common practice to formulate the structural analysis of complex structures as sets of linear or nonlinear algebraic equations. This is usually accomplished in one of three ways: 1) by using a discrete-element structural idealization, 2) by using a continuous idealization leading to partial differential equations that are then put in finite-difference form, 3) by using a continuous idealization in conjunction with assumed-mode stationary-energy methods. In each case, the formulation leads to a set of simultaneous algebraic equations.

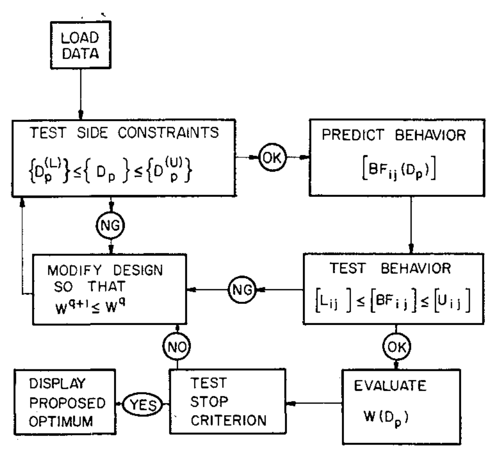

Fig. 1 Design cycle block diagram.

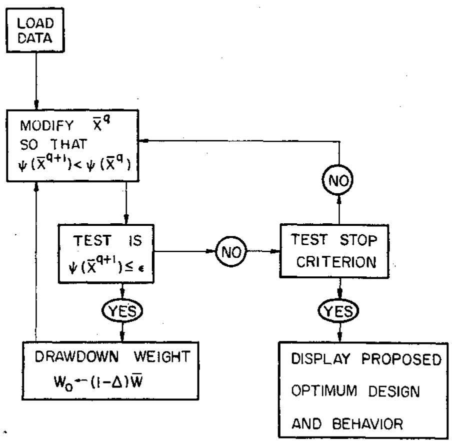

Fig. 2 Integrated synthesis-analysis block diagram.

It should be noted that the structural-analysis problem is basically nonlinear and that the common linearization rests on three assumptions that are far less likely to be satisfied as optimum-design capabilities improve. The three basic assumptions, which, when valid, permit linearization of the structural-analysis problem, may be stated as follows.

1) Displacements produce negligible changes in the basic geometry of the structure, and hence, force-equilibrium considerations may be based on the undeformed geometry of the structure.

2) Displacements and strains are small, and hence strain displacement relations are linear.

3) The force levels in the elements are sufficiently low to employ an ideal linear elastic stress-strain law.

Finding the solution of a system of simultaneous algebraic equations may be viewed as the problem of finding the minimum of a function that is made up of the square of the residuals of the algebraic system. For example, suppose that the equations representing the analysis of a structural system are

$$\begin{aligned} a_{11}\sigma_1 + a_{12}\sigma_2 &= c_1 P_1 \\ a_{21}\sigma_1 + a_{22}\sigma_2 &= c_2 P_1 \end{aligned} \tag{1}$$

and it is desired to find σ_1 and σ_2 where $a_{ij,ci}$, and P_1 are known. The σ_1 and σ_2 that satisfy Eq. (1) are those that make

$$\theta_1(\sigma_1, \sigma_2) = (a_{11}\sigma_1 + a_{12}\sigma_2 - c_1P_1)^2 + (a_{21}\sigma_1 + a_{22}\sigma_2 - c_2P_1)^2 \rightarrow 0 \tag{2}$$

Therefore, the requirement that the analysis equations be satisfied may be stated as a minimization problem as follows. Given a_{ij}, c_i, and P_1, find σ_1 and σ_2 such that $\theta_1(\sigma_1, \sigma_2) \rightarrow$ min. Note that this differs from the usual minimization problem because the minimum of θ_1 is known to be zero, and what is really desired are the values of σ_1 and σ_2 which make $\theta_1(\sigma_1, \sigma_2) \rightarrow 0$. This method of solving algebraic equation systems has a vast literature[2], but it is generally conceded that, in the case of linear equations, it is not the most efficient method. In the case of nonlinear equations, the efficiency of the method of residual minimization is an open question. In any event, it is clear that the method of residual minimization can be used to solve a system of algebraic equations.

Suppose now that the a_{ij} are assumed to be design variables and that θ_2 is the function θ_1 but with the a_{ij} variable. Then the requirement that the analysis equations be satisfied for any design may be stated as a minimization problem as follows. Given c_i and P_1, find σ_1, σ_2, and a_{ij} such that $\theta_2(\sigma_1, \sigma_2, a_{ij}) \rightarrow 0$. Any solution of this problem will be a design, described by the a_{ij}, and its analysis, described by σ_1 and σ_2. However, it is apparent that the number of sets of values a_{ij}, σ_j for which $\theta_2 = 0$ is infinite. If there are inequality constraints on the acceptable values of the a_{ij} and the σ_j, these can be included in a function

$$\theta_3(a_{ij}, \sigma_j) = \theta_2 + \sum_{j=1}^{2} \left[\langle LB_j - \sigma_j\rangle^2 + \langle \sigma_j - UB_j\rangle^2 + \sum_{i=1}^{2} \langle LB_{ij} - a_{ij}\rangle^2 + \sum_{i=1}^{2} \langle a_{ij} - UB_{ij}\rangle^2 \right] \tag{3}$$

where the angular brackets are the bracket functions

$$\langle Z\rangle^\circ = \begin{cases} 0 & Z \leq 0 \\ 1 & Z > 0 \end{cases} \tag{4}$$

$$\langle Z\rangle^n = Z^n \langle Z\rangle^\circ \qquad n > 0 \tag{5}$$

UB equals the upper bound on a given variable, and LB equals the lower bound on a given variable. The requirements that the analysis equations be satisfied, that the behavior variables σ_j lie between the given upper and lower bounds, and that the design variables lie between the given upper and lower bounds may be stated as a minimization problem as follows. Given c_i, P_1, LB_j, UB_j, LB_{ij}, and UB_{ij}, find σ_1, σ_2, and a_{ij} such that $\theta_3(\sigma_1, \sigma_2, a_{ij}) \rightarrow 0$, where θ_3 is given by Eq. (3). It is observed that $\theta_3 = 0$ yields an acceptable design and its analysis, but there are still, in general, an infinite number of solution sets (σ_j, a_{ij}). Note that the introduction of the "penalty functions" in forming θ_3 [Eq. (3)] does not affect the continuity of θ_3 or its first partial derivatives.

The weight of a structure can usually be expressed as a function of the design variables a_{ij}, that is,

$$W = W(a_{ij}) \tag{6}$$

The approach used in this paper to seek the minimum-weight design is to introduce an additional constraint

$$W(a_{ij}) \leq W_o \tag{7}$$

and incorporate it into a function

$$\psi = \langle W - W_o\rangle^2 + \theta_3 \tag{8}$$

where W_o, which will subsequently be called the draw-down weight, is a goal weight for the particular draw-down cycle and the angular brackets have the significance defined by Eqs. (4) and (5). Suppose W_o is selected greater than the minimum weight that can be achieved. Then the minimum of $\psi \rightarrow 0$, because each of the many designs satisfying $LB_{ij} \leq a_{ij} \leq UB_{ij}$ and having a weight W_o or less implies a set of behavior variables σ_j which satisfies the analysis equations; and, for some of these designs, $LB_j \leq \sigma_j \leq UB_j$ is satisfied. In other words, if the W_o selected is larger than the minimum weight that can be achieved, the minimum of $\psi = 0$, because there are numerous designs of weight W_o or less for which $\theta_3 = 0$. If W_o takes on a value less than the minimum weight that can be achieved, then the minimum of $\psi > 0$, and clearly this situation can be detected. The requirements that the analysis equations be satisfied, the behavior variables σ_j lie between the given upper and lower bounds, the design variables a_{ij} lie between the given upper and lower bounds, and the weight be less than or equal to the draw-down weight W_o may be stated as an unconstrained minimization problem as follows.

Given c_i, P_1, LB_j, UB_j, LB_{ij}, UB_{ij}, and W_o, find σ_1, σ_2, and a_{ij} such that $\psi(\sigma_1, \sigma_2, a_{ij}) = \langle W - W_o\rangle^2 + \theta_3 \rightarrow$ min, where θ_3 is given by Eq. (3). Any solution of this minimization problem for which the minimum of $\psi \rightarrow 0$ represents a design and the corresponding analysis for an acceptable design of weight W_o or less. If this weight is designated as $\bar{W}$, and now W_o is replaced by $W_o' = \bar{W} - \Delta\bar{W}$, and the minimum of $\psi \rightarrow 0$ again, an acceptable design at least $\Delta\bar{W}$ lb lighter has been achieved. Repeating this "draw-down" process eventually leads to a value of W_o' for which it is not possible to make $\psi \rightarrow 0$. This W_o' will then be below the achievable weight, and the preceding $\bar{W}$ will be above the achievable weight. Therefore, an acceptable design that is within $\Delta\bar{W}$ lb of the achievable minimum weight will have been obtained.

The applied mathematics literature abounds with techniques for solving unconstrained minimization problems. A selected bibliography on this subject is given in Appendix A of Ref. 2. Almost all methods rely upon the fact that, if a sequence of points $\{\mathbf{x}^q\}$ is generated such that $\psi(\mathbf{x}^{q+1}) < \psi(\mathbf{x}^q)$, a minimum will be reached in time. Many methods use the property of differentiable functions that the function decreases locally at the greatest rate in the direction of $-\nabla\psi$. That is, the sequence is generated as

$$\mathbf{x}^{q+1} = \mathbf{x}^q - h_q \nabla \psi(\mathbf{x}^q)$$

where h_q is a positive scalar constant. Considerable attention has been given in the literature to the question of what constants to use for h_q and how to prevent or make use of zigzagging. The scheme used to obtain the results for this paper is essentially the same as that given in Ref. 6. The only differences are that here the exact components of $\nabla\psi$ were used, and the number of moves before a zigzag move was taken was different. The method employed is also explained in more detail in Ref. 2.

The block diagram shown in Fig. 2 illustrates the integrated analysis-synthesis optimization process. A comparison of Fig. 2 with Fig. 1 reveals that, in the integrated approach, the design-cycle process is essentially eliminated. It is emphasized that, in the integrated approach, analyses are accomplished only for designs that are acceptable and which offer at least a specified weight reduction $\Delta\bar{W}$.

A means of proving that a proposed optimum design is indeed an absolute-minimum-weight design is not known at present. In this paper confidence that the proposed optimum designs presented are absolute optimum designs has been built by running multiple integrated synthesis-analysis paths from widely separated starting points.

Illustrative Example: *m*-Bar Truss

Using the concepts put forward in the previous section, the minimum-weight design of an m-bar planar truss, subject to n distinct load conditions as well as stress, displacement, and buckling limitations, is formulated as an unconstrained minimization problem. An integrated synthesis formulation of a general space truss is outlined in Appendix B of Ref. 2. The m-bar planar truss is shown in Fig. 3. Each load condition k, where $k = 1, \ldots, n$ is specified by giving the magnitude of the load P_k, the angle α_k measured clockwise from the positive x axis to the load P_k, and the temperature change in each member ΔT_{ik} for $i = 1, \ldots, m$. The cross-sectional areas $\bar{A}_i$ and the position of the attachment points $\bar{d}_i$ are the design variables. Note that the option to treat the $\bar{d}_i$ as preassigned design parameters exists.

The equilibrium equations representing the sum of the forces in the x and y directions at the node p', respectively, are

$$\sum_{i=1}^{m} \bar{A}_i\bar{\sigma}_{ik}\cos\beta_{ik} + P_k\cos\alpha_k = 0 \qquad k = 1, 2, \ldots, n \quad (9)$$

$$\sum_{i=1}^{m} \bar{A}_i\bar{\sigma}_{ik}\sin\beta_{ik} - P_k\sin\alpha_k = 0 \qquad k = 1, 2, \ldots, n \quad (10)$$

where $\bar{A}_i$ is the cross-sectional area of the ith member, β_{ik} is the angle between the ith member in the deformed state for the kth load condition and the positive x axis, and $\bar{\sigma}_{ik}$ is the stress in the ith member due to the kth load condition. From the geometry of Fig. 3, it follows that

$$\cos\beta_{ik} = (\bar{d}_i - \bar{u}_k)/[(H - \bar{v}_k)^2 + (\bar{d}_i - \bar{u}_k)^2]^{1/2} \quad (11)$$

$$\sin\beta_{ik} = (H - \bar{v}_k)/[(H - \bar{v}_k)^2 + (\bar{d}_i - \bar{u}_k)^2]^{1/2} \quad (12)$$

where $\bar{u}_k$ is the x component of the displacement of node p, $\bar{v}_k$ is the y component of the displacement of the node p, and H is a preassigned parameter giving the normal distance from node p to the support surface $r - r$ prior to deformation. The stresses $\bar{\sigma}_{ik}$ and the displacements $\bar{u}_k$, $\bar{v}_k$ are the behavior variables. Using the Ramberg-Osgood stress-strain relation the stress displacement equation for the ith member in the kth load condition is

$$\frac{\bar{\sigma}_{ik}}{E_i} + K_i\left(\frac{\bar{\sigma}_{ik}}{E_i}\right)^{M_i} + \bar{\alpha}_i\,\Delta T_{ik} = \frac{[(H - \bar{v}_k)^2 + (\bar{d}_i - \bar{u}_k)^2]^{1/2}}{(H^2 + \bar{d}_i^2)^{1/2}} - 1 \quad (13)$$

$$i = 1, 2, \ldots, m \qquad k = 1, 2, \ldots, n$$

where

$$K_i = \tfrac{3}{7}(E_i/Y_i)^{M_i - 1} \quad (14)$$

Y_i is the yield stress, and $\bar{\alpha}_i$ is the thermal-expansion coefficient for the material of the ith member. The exponent M_i depends on the material used for the ith member, and it is convenient to restrict M_i to positive odd-integer values for the present purposes.

The physical variables are $\bar{A}_i$, $\bar{\sigma}_{ik}$, $\bar{u}_k$, $\bar{v}_k$, and $\bar{d}_i$. However, the ψ function may have better properties if the variables on which it depends are scaled so as to be of the order of ± 1. This is accomplished by introducing dimensionless variables as follows:

$$\bar{A}_i = K_A A_i \quad (15)$$

$$\bar{\sigma}_{ik} = K_\sigma \sigma_{ik} \quad (16)$$

$$\bar{u}_k = K_u u_k \quad (17)$$

$$\bar{v}_k = K_u v_k \quad (18)$$

$$\bar{d}_i = H d_i \quad (19)$$

where the K_A, K_σ, and K_u are constants equal to the expected average order of magnitude of the areas, stresses, and displacements, respectively.

Let the following notations be introduced:

$$PX_k = +P_k\cos\alpha_k \quad (20)$$

$$PY_k = -P_k\sin\alpha_k \quad (21)$$

$$\cos\beta_{ik} = LDX_{ik}/LD_{ik} \quad (22)$$

$$\sin\beta_{ik} = LDY_{ik}/LD_{ik} \quad (23)$$

$$LDX_{ik} = d_i - (K_u/H)u_k = \overline{LDX}_{ik}/H \quad (24)$$

$$LDY_k = 1 - (K_u/H)v_k = \overline{LDY}_{ik}/H \quad (25)$$

$$LD_{ik} = \{[1 - (K_u/H)v_k]^2 + [d_i - (K_u/H)u_k]^2\}^{1/2} \quad (26)$$

$$C_{ik} = K_A K_\sigma \cos\beta_{ik} \quad (27)$$

$$S_{ik} = K_A K_\sigma \sin\beta_{ik} \quad (28)$$

If both of the equilibrium equations [Eqs. (9) and (10)] are divided by $\bar{P}$, the average magnitude of the nonzero applied

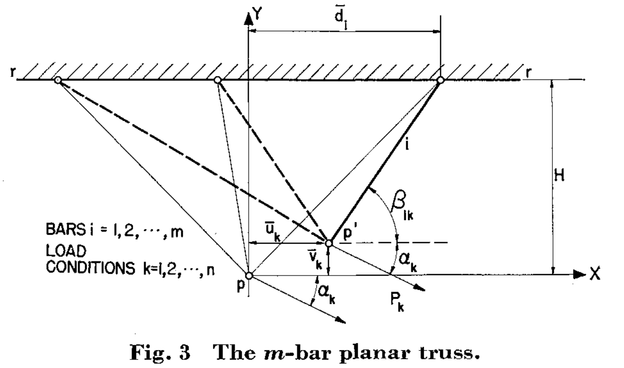

Fig. 3 The *m*-bar planar truss.

loads P_k, then the functions EQX_k and EQY_k may be defined as follows:

$$EQX_k(A_i, \sigma_{ik}, u_k, v_k, d_i) = \left(\sum_{i=1}^{m} A_i\sigma_{ik}C_{ik} + PX_k\right)\Big/\bar{P} \qquad k = 1, 2, \ldots, n \quad (29)$$

$$EQY_k(A_i, \sigma_{ik}, u_k, v_k, d_i) = \left(\sum_{i=1}^{m} A_i\sigma_{ik}S_{ik} + PY_k\right)\Big/\bar{P} \qquad k = 1, 2, \ldots, n \quad (30)$$

These two functions represent the fraction of the average applied load by which each of the equilibrium equations is unsatisfied in the kth load condition. If the dimensionless design variables A_i, d_i, and the dimensionless behavior variables σ_{ik}, u_k, v_k satisfy equilibrium, then

$$EQX_k = EQY_k = 0 \qquad k = 1, 2, \ldots, n$$

The stress displacement equations may be treated in a similar manner. Let the following notation be introduced:

$$R_i = \tfrac{3}{7}\,[K_\sigma/Y_i]^{M_i - 1} \quad (31)$$

$$\eta_i = E_i/K_\sigma \quad (32)$$

$$\tau_{ik} = \bar{\alpha}_i E_i \Delta T_{ik}/K_\sigma \quad (33)$$

$$LO_i = [1 + d_i^2]^{1/2} = \overline{LO}_i/H \quad (34)$$

$$\tilde{P} = \bar{P}/K_A K_\sigma \quad (35)$$

Now multiplying Eq. (13) by $E_i\bar{A}_{io}$, where $\bar{A}_{io}$ is a constant of the order $\bar{A}_i$, and dividing by $\bar{P}$ leads to the following definition of the function SD_{ik}:

$$SD_{ik}(A_i, \sigma_{ik}, u_k, v_k, d_i) = (A_{io}/\tilde{P})[\sigma_{ik} + R_i\sigma_{ik}^{M_i} + \tau_{ik} + \eta_i(1 - LD_{ik}/LO_i)]$$
$$i = 1, 2, \ldots, m \qquad k = 1, 2, \ldots, n \quad (36)$$

This function approximates the fraction of the average applied load by which the two representations of the force in member i for load condition k differ. Therefore, sets of variables (A_i, σ_{ik}, u_k, v_k, d_i) that satisfy the analysis equations have the property that

$$EQX_k = 0 \qquad k = 1, 2, \ldots, n$$
$$EQY_k = 0 \qquad k = 1, 2, \ldots, n$$
$$SD_{ik} = 0 \qquad i = 1, 2, \ldots, m;\; k = 1, 2, \ldots, n$$

The weight of the m-bar planar truss may be expressed as follows:

$$W = \sum_{i=1}^{m} (H^2 + \bar{d}_i^2)^{1/2}\rho_i\bar{A}_i \quad (37)$$

where ρ_i is the weight density of the material for the ith member. Let the following notation be introduced:

$$\delta_i = \rho_i H K_A/W_o$$
$$W_o = \text{draw-down weight} \quad (38)$$

Then a function $FW(A_i, d_i)$ that represents the fraction of W_o by which the weight W exceeds W_o is defined as follows:

$$FW(A_i, d_i) = \left\langle \frac{W - W_o}{W_o} \right\rangle^1 = \left\langle \sum_{i=1}^{m} \delta_i LO_i A_i - 1 \right\rangle^1 \quad (39)$$

In general, there are upper and lower bounds on all the variables A_i, σ_{ik}, u_k, v_k, d_i. Let the set of variables be represented by x_j, where

$$x_j = A_i \qquad j = i \qquad i = 1, 2, \ldots, m \quad (40)$$

$$x_j = \sigma_{ik} \qquad j = mk + i \qquad i = 1, 2, \ldots, m;\; k = 1, 2, \ldots, n \quad (41)$$

$$x_j = u_k \qquad j = m(1 + n) + 2k - 1 \qquad k = 1, 2, \ldots, n \quad (42)$$

$$x_j = v_k \qquad j = m(1 + n) + 2k \qquad k = 1, 2, \ldots, n \quad (43)$$

$$x_j = d_i \qquad j = m(1 + n) + 2n + i \qquad i = 1, 2, \ldots, m \quad (44)$$

Note that, for a complete set of variables, $j = 1, 2, \ldots, l$, and

$$l = 2(m + n) + mn \quad (45)$$

A function providing a penalty for violating an upper or a lower bound for each variable x_j may now be defined as follows:

$$BND_j = \langle LB_j - x_j\rangle^1 + \langle x_j - UB_j\rangle^1 \qquad j = 1, 2, \ldots, l \quad (46)$$

where LB_j and UB_j are the lower and upper bounds on the variables x_j, respectively.

In addition to simple bounds, provision is made for limiting the stress in order to preclude buckling. The Euler-Engesser, or tangent modulus, buckling stress constraint may be expressed as follows§:

$$TB_{ik} = \langle G_{ik}A_i - \sigma_{ik}\rangle^1 \quad (47)$$

where

$$G_{ik} = -\frac{1}{SF_{ik}}\,\frac{\pi K_A}{8H^2(LO_i)^2}\,\frac{\eta_i}{[1 + M_iR_i\,\sigma_{ik}^{(M_i-1)}]} \times \left[\left(\frac{D}{t}\right)_i + \frac{1}{(D/t)_i}\right] \quad (48)$$

SF_{ik} is a buckling stress safety factor for the ith member in the kth load condition, and $(D/t)_i$ is the mean diameter-to-thickness ratio for the ith member.

The minimum-weight design of an m-bar planar truss (Fig. 3), subject to n distinct load conditions as well as stress, displacement, and buckling limitations, can be accomplished by solving the following problem for a sequence of decreasing values of W_o. Given the preassigned parameters $(D/t)_i$, SF_{ik}, H, M_i, Y_i, E_i, $\bar{\alpha}_i$, $\bar{P}_k$, α_k, ΔT_{ik}, a set of initial values $\bar{A}_i$, $\bar{\sigma}_{ik}$, $\bar{u}_k$, $\bar{v}_k$, $\bar{d}_i$, W_o, and a set of bounds LB_j, UB_j, find $\bar{A}_i$, $\bar{\sigma}_{ik}$, $\bar{u}_k$, $\bar{v}_k$, $\bar{d}_i$ such that $\psi(A_i, \sigma_{ik}, u_k, v_k, d_i) \to \min$, where ψ is given by

$$\psi = \lambda_w(FW)^2 + \sum_{k=1}^{n}\left[EQX_k^2 + EQY_k^2 + \sum_{i=1}^{m}(SD_{ik})^2\right] + \sum_{j=1}^{l}\lambda_j BND_j^2 + \sum_{k=1}^{n}\sum_{i=1}^{m}\lambda_{TB_{ik}}(TB_{ik})^2 \quad (49)$$

and the λ's are multipliers to be discussed subsequently. It is apparent that any set of values for the variables A_i, σ_{ik}, u_k, v_k, and d_i, for which $\psi = 0$, describes an acceptable design A_i, d_i of weight W_o or less and behavior σ_{ik}, u_k, v_k. It is again remarked that ψ and $\nabla\psi$ are continuous.

The method used to find $\psi \to 0$ will be numerical and, therefore, a criterion is needed to determine when ψ is near enough to zero to be considered converged. From the way that the functions EQX_k, EQY_k, and SD_{ik} are defined, it is clear that an ϵ may be chosen such that the requirement

$$\sum_{k=1}^{n}[EQX_k^2 + EQY_k^2] + \sum_{i=1}^{m}\sum_{k=1}^{n} SD_{ik}^2 \le \epsilon$$

will be meaningful in terms of the physical problem. For example, $\epsilon = 0.0001$ insures that, at worst, an equilibrium or

§ An annular section is assumed for each member.

force-displacement equation will be unsatisfied by 1% of the average force level applied to the structure. It can, of course, be chosen smaller, but it should be noted that, even in a linear problem where the analysis solution is obtained by matrix inversion, the residuals are seldom truly zero as a result of roundoff noise and other factors.[7] It is suggested that this method offers a controlled and physically significant criterion for analysis accuracy.

The λ's must now be chosen so that, when $\psi \leq \epsilon$, the various constituents of ψ will be satisfied meaningfully. Let each part of ψ be treated as if it were the only contributor to the value of ψ. Consider first the fractional weight-function term

$$\lambda_w \langle W/W_o - 1\rangle^2 \leq \epsilon \quad (50)$$

The λ_w is to be selected so that this criterion imposes what is necessary, while at the same time not being so stringent as to unnecessarily delay convergence. It is required that the weight W not exceed the draw-down weight W_o by more than a small fraction of the incremental weight decrease ΔW_o. In other words, it is required that if $W - W_o > 0$ then

$$W - W_o \ll \Delta W_o \quad (51)$$

Now if s is an assigned number, small compared with unity, then it follows from Eq. (51) that

$$W/W_o - 1 = s\Delta \quad (52)$$

and substituting into Eq. (50) and taking the equality yields

$$\lambda_w = \epsilon/s^2\Delta^2 \quad (53)$$

Use of the value for λ_w given by Eq. (53) insures that, when $\psi \leq \epsilon$,

$$W/W_o - 1 \leq s\Delta \quad (54)$$

since $\lambda_w\langle W/W_o - 1\rangle^2 \leq \epsilon$ implies $(\epsilon/s^2\Delta^2)\langle W/W_o - 1\rangle^2 \leq \epsilon$, which reduces to Eq. (54) if $W - W_o > 0$.

The determination of the λ_j's associated with the simple bounds can be treated in a similar fashion. Let Q_j be the amount by which the dimensionless variables x_j may be permitted to violate the bound LB_j or UB_j. Assume that the upper bound UB_j happens to be the active bound; then it is required that

$$\lambda_j(x_j - UB_j)^2 \leq \epsilon \quad (55)$$

must insure that

$$x_j - UB_j \leq Q_j \quad (56)$$

Substituting Eq. (56) into Eq. (55) yields

$$\lambda_j(Q_j)^2 \geq \epsilon \quad (57)$$

Taking the equality and solving for λ_j gives

$$\lambda_j = \epsilon/Q_j^2 \quad (58)$$

It follows that, when $\psi \leq \epsilon$, the value of x_j will never exceed $UB_j + Q_j$ nor be less than $LB_j - Q_j$.

The determination of the $\lambda_{TB_{ik}}$ associated with the buckling stress limitations can also be treated in a similar manner. Let g be the fraction of the dimensionless allowable buckling stress by which the dimensionless allowable stress $G_{ik}A_i$ and the dimensionless stress σ_{ik} may differ. Then it is required that

$$\lambda_{TB_{ik}}(G_{ik}A_i - \sigma_{ik})^2 \leq \epsilon \quad (59)$$

insure that

$$G_{ik}A_i - \sigma_{ik} \leq G_{ik}A_i g \quad (60)$$

Substituting Eq. (60) into Eq. (59) yields

$$\lambda_{TB_{ik}}(G_{ik}A_i g)^2 \geq \epsilon \quad (61)$$

Taking the equality and solving for $\lambda_{TB_{ik}}$ gives

$$\lambda_{TB_{ik}} = \epsilon/(G_{ik}A_i g)^2 \quad (62)$$

Thus the numbers s, Δ, Q_j, g, and ϵ must be selected in order to construct the ψ function. This is no particular burden as these quantities all have a clear engineering significance in this context.

The ψ function for the m-bar planar truss with a linearized analysis can be obtained by specialization of the foregoing formulation. The parts of the ψ function effected by linearization are EQX_k, EQY_k, SD_{ik}, and TB_{ik}. The influence of linearizing the analysis on the functions EQX_k and EQY_k leaves Eqs. (29) and (30) unchanged; however, Eqs. (27) and (28) are modified to read

$$C_i = K_A K_\sigma \cos\beta_i \quad (27a)$$

and

$$S_i = K_A K_\sigma \sin\beta_i \quad (28a)$$

where β_i is the angle between the ith member and the positive x axis in the undeformed state. This simplification follows from the assumption that the equilibrium equations can be based on the undeformed geometry of the structure. The influence of linearizing the analysis on the functions SD_{ik} can be obtained by setting R_i equal to zero in Eq. (36) and noting that

$$\left[1 - \frac{LD_{ik}}{LO_i}\right] = \frac{(K_u/H)[2(v_k + d_i u_k) - (K_u/H)(v_k^2 + u_k^2)]}{LO_i\,(LO_i + LD_{ik})} \quad (63)$$

For small displacements,

$$(K_u/H)(v_k^2 + u_k^2) \ll 2(v_k + d_i u_k) \quad (64)$$

and, for small deformations,

$$LO_i(LO_i + LD_{ik}) \approx 2(LO_i)^2 \quad (65)$$

Therefore,

$$[1 - (LD_{ik}/LO_i)] \approx (K_u/H\,LO_i)\,(v_k \sin\beta_i + u_k \cos\beta_i) \quad (66)$$

Thus the resulting expression for SD_{ik}, based on a linearized analysis, is

$$SD_{ik} = (A_{io}/\bar{P})[\sigma_{ik} + \tau_{ik} + \eta_i(K_u/H\,LO_i) \times (v_k \sin\beta_i + u_k \cos\beta_i)] \quad (67)$$

The influence of linearizing the analysis on the function TB_{ik} is obtained by setting R_i equal to zero in Eq. (48). The buckling stress limits are then based on Euler buckling stress, rather than on the tangent modulus buckling stress.

In the next section, minimum-weight optimum designs for various m-bar planar-truss problems, obtained using an integrated synthesis-analysis approach, are presented.

Results and Discussion

In this section, numerical results are presented for several example problems. These results have been obtained using a digital-computer program based on the ψ function for an m-bar planar truss and the minimization technique previously mentioned.

Consider a three-bar aluminum truss subject to three distinct load conditions in which the configuration is fixed by preassigning $\bar{d}_1 = -20$ in., $\bar{d}_2 = 0$, $\bar{d}_3 = +20$ in., and $H = 20$ in. The three load conditions are given in Table 1. Since the members are to be made of aluminum alloy, $E_i = 10.4 \times 10^3$ ksi, $\bar{\alpha}_i = 12.5 \times 10^{-6}$ in./in.-°F, and $\rho_i = 0.101$ lb/in.^{3}. The ratio $(D/t)_i$ is assigned a value of 10. Furthermore, the buckling stress safety factors are set equal to unity. The stress limits are taken to be ±71.5 ksi, and the displacement limits are taken to be ±0.2 in. on both $\bar{u}_k$ and $\bar{v}_k$ (see Table 2).

Table 1 Load conditions for cases 1 and 2

Load condition, k	P_k, kip	α_k, deg	ΔT_{1k}, °F	ΔT_{2k}, °F	ΔT_{3k}, °F
1	100	0	50	100	150
2	135	135	150	100	50
3	100	35	0	0	0

The only difference between case 1 and case 2 is that case 1 is based on a linear analysis and case 2 employs a nonlinear analysis. For case 1 then, the only additional data are the values of the ψ-function control parameters, and they are $\epsilon = 0.0001$, $s = 0.1$, $\Delta = 0.05$, and $g = 0.01$. The results for case 1 are shown in Table 2, and they compare well with previous results[8] obtained using a design-cycle-based method. The previously reported result is $A_1 = 1.124$ in.,^{2} $A_2 = 0.523$ in.^{2}, $A_3 = 1.610$ in.^{2}, and $W = 8.869$ lb.

For case 2, the ψ-function control parameters are the same as those used for case 1; however, the nonlinear analysis is retained, and this requires the additional data $Y = 71.5$ ksi and $M_i = 11$. The results for case 2 are also shown in Table 2, and they may be compared with those obtained in case 1 using a linearized analysis. It is interesting to note that in case 2 the displacements are generally larger, which is not surprising. In particular, it is seen that the stresses $\bar{\sigma}_{31}$ and $\bar{\sigma}_{32}$ are close to their limiting values in case 1, whereas $\bar{\sigma}_{31}$ and $\bar{u}_3$ are nearly bound in case 2. Note that the minimum-weight design achieved, based on the nonlinear analysis, is slightly heavier than that obtained when the analysis is linearized. This is reasonable, since the linearized analysis was based on the initial tangent modulus, which implies a stiffness that is not actually present at higher stress levels.

Next consider a three-bar aluminum truss subject to four distinct load conditions for which $H = 20$ in. and the $\bar{d}_i$ and $\bar{A}_i$ are design variables. The four load conditions are given in Table 3. Note that load conditions 2 and 4 are obtained from load conditions 1 and 3, respectively, by multiplying the magnitude of the loads by the not unfamiliar factor 1.5. The material properties used in case 3 are $E_i = 10 \times 10^3$ ksi, $\rho_i = 0.1$ lb/in.^{3}, $Y_i = 45$ ksi, and $N_i = 11$. The ratio $(D/t)_i$ is assigned a value of 20. The buckling stress safety factors for load conditions 1 and 3 are set equal to 1.5, and those for load conditions 2 and 4 are set equal to unity. The stress limits for load conditions 1 and 3 are taken to be ± 45 ksi, and for load conditions 2 and 4 they are taken to be ± 65 ksi (see Table 4). The displacement limits for load conditions 1 and 3 are set equal to ± 0.2 in., and for load conditions 2 and 4 they are taken to be ± 2.0 in. (see Table 4). The results for case 3 are shown in Table 4 and were obtained using the same ψ-function control parameters as in cases 1 and 2. Note that the displacements are all small and that the stress $\bar{\sigma}_{12}$ is critical in buckling.

Some comments on the running times for these results are in order. Figure 4 is a graphical presentation of the running-time breakdown for the three cases reported in this paper. The time spent in solving the synthesis problem for the integrated method may be divided into three periods: initial convergence, synthesis, and termination.

The initial convergence period is spent in obtaining an analysis and design that are satisfactory and of weight W_o or less. If a design and its correct analysis are known at the outset, and if W_o is set equal to the weight of that design, the initial convergence period is eliminated entirely. In the case presented in this paper the starting values were not of this type. In the linear case (case 1) it is clear that, by the simple inversion of a 5×5 matrix, the initial analysis could be obtained and the 2.7 sec¶ of initial convergence essentially eliminated. This was not done here, because it was desirable to test the penalty involved in not doing so. The initial values chosen were rather poor, some being of the wrong sign.

In the nonlinear cases (cases 2 and 3) an initial analysis is not as easy to obtain. One approach would be to start with the results of a linear analysis and complete the initial convergence. This would probably speed up the process considerably. Case 2, which is a nonlinear version of case 1, was started from exactly the same point as was case 1, and the initial convergence required 4.5 sec. All of the increase in running time is attributed to the increased computational time required to evaluate the ψ function and its derivatives for the nonlinear formulation, since essentially the same number of cycles through the minimizer routine were re-

Table 2 Numerical results, cases 1 and 2

j	Load condition, k		LB_j	UB_j	Q_j	Case 1 Kx_j Start	Case 1 Kx_j Final	Case 2 Kx_j Start	Case 2 Kx_j Final
1	N/A	$\bar{A}_1$	0.02	3.0	0.01	1.6	1.171	1.6	1.176
2		$\bar{A}_2$	0.02	3.0	0.01	1.5	0.540	1.5	0.484
3		A_3	0.02	3.0	0.01	2.0	1.573	2.0	1.635
4	1	$\bar{\sigma}_{11}$	−70.8	70.8	0.7	50.00	54.43	50.00	54.32
5		$\bar{\sigma}_{21}$	−70.8	70.8	0.7	50.00	17.73	50.00	21.07
6		$\bar{\sigma}_{31}$	−70.8	70.8	0.7	−10.00	−49.12*	−10.00	−47.36*
7	2	$\bar{\sigma}_{12}$	−70.8	70.8	0.7	−10.00	−20.61	−10.00	−19.29
8		$\bar{\sigma}_{22}$	−70.8	70.8	0.7	50.00	62.91	50.00	66.76
9		$\bar{\sigma}_{32}$	−70.8	70.8	0.7	50.00	70.48*	50.00	68.46
10	3	$\bar{\sigma}_{13}$	−70.8	70.8	0.7	50.00	68.25	50.00	66.75
11		$\bar{\sigma}_{23}$	−70.8	70.8	0.7	50.00	46.30	50.00	56.85
12		$\bar{\sigma}_{33}$	−70.8	70.8	0.7	−10.00	−22.34	−10.00	−22.16
13	1	$\bar{u}_1$	−0.198	0.198	0.002	0.100	0.174	0.100	0.174
14		$\bar{v}_1$	−0.198	0.198	0.002	0.100	−0.060	0.100	−0.065
15	2	$\bar{u}_2$	−0.198	0.198	0.002	0.100	−0.150	0.100	−0.181
16		$\bar{v}_2$	−0.198	0.108	0.002	0.100	−0.146	0.100	−0.180
17	3	$\bar{u}_3$	−0.198	0.198	0.002	0.100	0.175	0.100	0.196*
18		$\bar{v}_3$	−0.198	0.198	0.002	0.100	−0.089	0.100	−0.113
19	N/A	$\bar{d}_1$				−20.0	−20.0	−20.0	−20.0
20		d_2	N/A	N/A	N/A	0	0	0	0
21		$\bar{d}_3$				+20.0	+20.0	+20.0	+20.0
		$\bar{W}$				N/A	8.934	N/A	9.005
		W_o				14.00	8.921	14.00	8.980

¶ All times quoted in this section are running times on a Univac 1107 computer.

Table 3 Load conditions for case 3[a]

Load condition, k	P_k, kip	α_k, deg
1	100	0
2	150	0
3	70	270
4	105	270

[a] $\Delta T_{ik} = 0;\ i = 1, 2, 3;\ k = 1, 2, 3, 4.$

quired in both cases. Case 3 required a rather long time for initial convergence. This was probably because of a poor starting point.

The synthesis period is shown in Fig. 4 with a graph of the weight vs running time. It should be noted that, although there are smooth curves drawn here, the weight actually comes down in 5% increments. For example, at the initial convergence for case 2, a satisfactory design weighing 14 lb was known. When the first weight-reduction cycle was in progress, no new design was known for 2.5 sec, at which time the second convergence at a design of 13.3 lb was obtained. The curves were drawn through the points representing these discrete convergences. These plots exhibit a characteristic feature of most synthesis processes, that of diminishing returns. For example, in case 2 if the synthesis had been stopped after 25 sec, all but the last 5% reduction would have been obtained, and yet it took 25 sec more to get that last 5%. This characteristic is an important consideration if the task of weight reduction has a cost restriction.

The termination period is the length of time it takes to be convinced that it is not possible to obtain another 5% weight reduction. In a sense, the duration of this period is arbitrary, because some people are easier to convince than others. In the operating program the end of the termination period automatically occurred in any one of three ways: a) if $|\nabla \psi|$ approached zero without $\psi \rightarrow 0$, b) if no significant move in the gradient direction could be found which resulted

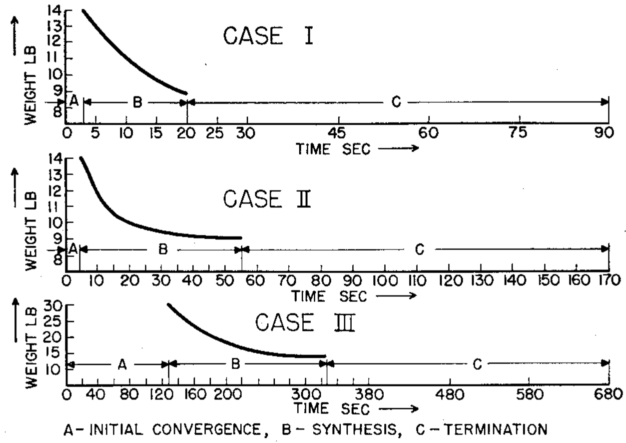

Fig. 4 Running time breakdown.

in a reduction in ψ, and ψ was not zero, or c) if after 2000 cycles through the minimizer ψ was not zero. Cases 1, 2, and 3 terminated in criteria a, b, and c, respectively. Criterion a is actually arbitrary, because the answer to the question "How small is zero?" is arbitrary. Criterion b is not arbitrary because, when the move size is repeatedly reduced until it loses numerical significance in the computer, there is nothing to do but stop. Criterion c is imposed because it may take an enormous number of cycles to satisfy a or b. It seems reasonable to assume that, if the largest number of cycles needed to converge a single weight draw-down on a given run is, say, 200, then a draw-down requiring more than 2000 cycles will probably not converge. The cutoff number may be moved up or down depending upon the required degree of certainty that another 5% reduction cannot be obtained. This question has, in fact, remained a difficult philosophical point in synthesis work. In some applications it is sufficient to obtain a substantial weight

Table 4 Numerical results, case 3

j	Load condition, k		LB_j	UB_j	Q_j	Case 3 Kx_j Start	Case 3 Kx_j Final
1	N/A	$\bar{A}_1$	0.01	4.0	0.01	3.00	0.887
2		$\bar{A}$	0.01	4.0	0.01	4.00	1.92
3		$\bar{A}$	0.01	4.0	0.01	4.00	2.65
4	1	$\bar{\sigma}_{11}$	−44.55	44.55	0.45	−5.00	−26.8
5		$\bar{\sigma}_{21}$	−44.55	44.55	0.45	0.00	−27.4
6		$\bar{\sigma}_{31}$	−44.55	44.55	0.45	5.00	28.8
7	2	$\bar{\sigma}_{12}$	−64.35	64.35	0.65	−7.00	−40.2*
8		$\bar{\sigma}_{22}$	−64.35	64.35	0.65	0.00	−40.9
9		$\bar{\sigma}_{32}$	−64.35	64.35	0.65	7.00	43.2
10	3	$\bar{\sigma}_{13}$	−44.55	44.55	0.45	−4.00	−17.6
11		$\bar{\sigma}_{23}$	−44.55	44.55	0.45	−4.00	−15.9
12		$\bar{\sigma}_{33}$	−44.55	44.55	0.45	−4.00	−17.6
13	4	$\bar{\sigma}_{14}$	−64.35	64.35	0.65	−5.00	−26.4
14	4	$\bar{\sigma}_{24}$	−64.35	64.35	0.65	−5.00	−23.9
15		$\bar{\sigma}_{34}$	−64.35	64.35	0.65	−5.00	−26.3
16	1	$\bar{u}$	−0.198	+0.198	0.002	0.100	0.114
17		$\bar{v}$	−0.198	+0.198	0.002	0.100	−0.003
18	2	$\bar{u}$	−1.98	+1.98	0.02	0.100	0.208
19		$\bar{v}$	−1.98	+1.98	0.02	0.100	−0.014
20	3	$\bar{u}$	−0.198	+0.198	0.002	0.100	−0.002
21		$\bar{v}$	−0.198	+0.198	0.002	0.100	0.059
22	4	$\bar{u}$	−1.98	+1.98	0.02	0.100	−0.003
23		$\bar{v}$	−1.98	+1.98	0.02	0.100	0.089
24	N/A	$\bar{d}$	−200.0	+200.0	2.0	0.00	16.1
25		$\bar{d}$	−200.0	+200.0	2.0	20.0	18.0
26		$\bar{d}$	−200.0	+200.0	2.0	−25.0	−17.13
		$\bar{W}$				N/A	14.43
		W_0				30.0	14.41

reduction when compared with what can be obtained by other means; in other applications it may be required to find the optimum design.

Confidence in the results presented has been built by running two complete integrated synthesis-analysis paths from widely separated starting points. These numerical results indicate the feasibility of the basic ideas put forward in the earlier sections of this paper.

The results of a preliminary investigation of a new approach to structural synthesis and analysis have been presented. The integrated analysis-synthesis concept is characterized by the fact that the quest for an improved design and the search for acceptable values of the behavior variables are carried on simultaneously. The usual design-cycle process, involving many unacceptable trial designs, is essentially eliminated, since analyses are accomplished only for designs that are acceptable and that offer at least a specified weight reduction. The integrated approach yields a formulation such that many available methods for finding the unconstrained minimum of a function of several variables can be brought to bear on the structural-synthesis problem. It has been shown that the integrated approach is well suited to seeking an answer to the feasibility type of question; namely, is a structural system (for which a ψ function has been developed) of weight W_o achievable? Furthermore, it has been shown that the integrated approach may be used to obtain a sequence of acceptable designs, each offering a specified weight reduction, terminating with a proposed minimum-weight optimum design and its behavior. Confidence that a proposed optimum design is a global minimum can be built by running multiple integrated synthesis-analysis paths from widely separated starting points.

Limited experience, with the m-bar single-node planar-truss computer program, indicates that the integrated approach offers the prospect of making substantial improvements in the efficiency of the structural-synthesis process, particularly when linearization of the structural analysis is inappropriate.

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